

All Static Circularly Symmetric Perfect Fluid Solutions of 2+1 Gravity

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Via a straightforward integration of the Einstein equations with cosmological constant, all static circularly symmetric perfect fluid 2+1 solutions are derived. The structural functions of the metric depend on the energy density, which remains in general arbitrary. Spacetimes for fluids fulfilling linear and polytropic state equations are explicitly derived; they describe, among others, stiff matter, monatomic and diatomic ideal gases, nonrelativistic degenerate fermions, incoherent and pure radiation. As a by-product, we demonstrate the uniqueness of the constant energy density perfect fluid within the studied class of metrics. A full similarity of the perfect fluid solutions with constant energy density of the 2+1 and 3+1 gravities is established.

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I. INTRODUCTION

In the last two decades a number of researches has been developed in 2+1 gravity: the search of exact solutions, the quantization of fields coupled to gravity, topological aspects, black hole physics, and so on. The literature on this respect is extremely vast. In the framework of exact solutions, from the beginning till now, the interest has been focused in the search and the study of physically relevant solutions and models with sources, for instance: static N -body spaces [1, 2], static and stationary metrics coupled to electromagnetic fields [3, 4, 5, 6, 7, 8], scalar—dilaton fields [9, 10], cosmology [11], perfect fluids [12, 13, 14], among others.

The purpose of this paper is the determination of all static circularly symmetric spacetimes with cosmological constant coupled to perfect fluids with and without zero pressure surfaces. In this context, some progress has been previously achieved. Cornish and Frankel [13] derived all universes obeying a polytropic equation. Nevertheless, since Cornish's solutions were derived for zero cosmological constant, there is no way to determine a zero pressure surface. Hence these solutions extend to the whole spacetime, and consequently they are cosmological solutions. On the other hand, Cruz and Zanelli [14] established some consequences arising from the hydrostatic equilibrium Oppenheimer–Volkov equation, and derived for this equation a single solution for constant energy density; it should be pointed out that the expression of their g_{tt} -metric component presents a misprint, which is corrected in this work. By the way, we demonstrate here that the perfect fluid solution with constant energy density is the only conformally flat—in the sense of the vanishing of the Cotton tensor [15]—circularly symmetric solution.

In Sec. II, by a straightforward integration of the Ein-

stein equations, we derive the general solution for the static circularly symmetric 2+1 metric with a cosmological constant coupled to a perfect fluid solution with variable density ρ and pressure p .

Sec. III is devoted to represent all this class of spacetimes in a canonical coordinate system. For a given equation of state of the form $p = p(\rho)$, certain particular families of perfect fluid solutions are derived; as concrete examples, the subcases of fluids obeying the linear law $p = \gamma\rho$, and those fluids subjected to a polytropic law $p = C\rho^\gamma$ in details are derived.

In Sec. IV, from the Oppenheimer–Volkov equation certain properties of the studied solutions are established, for instance, for positive pressure p and positive density ρ , obeying a state equation $p = p(\rho)$, a microscopically stable fluid possesses a monotonically decreasing energy density, and conversely.

In Sec. V, to facilitate the comparison of the interior Schwarzschild 3+1 solution with cosmological constant, the perfect fluid 2+1 solution with $\rho = \text{const.}$ is derived. With this aim in mind, we search for an adequate representation of the corresponding structural functions and related quantities of these 3+1 and 2+1 spacetimes. A comparison table is presented. Via a dimensional reduction of the interior Schwarzschild with λ solution, the perfect fluid 2+1 solution with constant ρ is obtained.

Finally, we end with some concluding remarks.

A. Einstein equations for 2+1 static circularly symmetric perfect fluid metric

As far as we know, in most of the publications dealing with the search of perfect fluid solutions in (2+1) gravity, see for instance [12, 13, 14], the energy–momentum conservation, i.e., the Oppenheimer–Volkov equation, has been used as a clue to obtain the desired results. On the contrary, we prefer to solve directly the corresponding Einstein equations; in such case the energy–momentum conservation equations trivially hold.

The line element of static circularly symmetric 2+1

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spacetimes, in coordinates $\{t, r, \theta\}$, is given by

$$ds^2 = -N(r)^2 dt^2 + \frac{dr^2}{G(r)^2} + r^2 d\theta^2. \quad (1)$$

Note that we are using units such that the velocity of light $c = 1$.

The Einstein equations with cosmological constant for a perfect fluid energy-momentum tensor T_{ab} :

$$\begin{aligned} G_{ab} &= R_{ab} - \frac{1}{2}g_{ab}R = \kappa T_{ab} - \lambda g_{ab}, \\ T_{ab} &= (p + \rho)u_a u_b + p g_{ab}, \quad u_a = -N\delta_a^t, \\ R &= 2\kappa\rho - 4\kappa p + 6\lambda, \end{aligned} \quad (2)$$

for the metric (1) explicitly amount to

$$\begin{aligned} G_{tt} &= -\frac{N^2}{2r} \frac{dG^2}{dr} = N^2(\kappa\rho + \lambda), \\ G_{rr} &= \frac{1}{rN} \frac{dN}{dr} = \frac{1}{G^2}(\kappa p - \lambda), \\ G_{\theta\theta} &= \frac{r^2}{N} \left[G^2 \frac{d^2 N}{dr^2} + \frac{1}{2} \frac{dN}{dr} \frac{dG^2}{dr} \right] \\ &= r^2(\kappa p - \lambda). \end{aligned} \quad (3)$$

Notice that the combination of the Einstein equations $r^2 G^2 G_{rr} - G_{\theta\theta} = 0$, for $N(r) \neq \text{const.}$, gives rise to an important equation:

$$\begin{aligned} G \frac{d^2 N}{dr^2} + \frac{dN}{dr} \left(\frac{dG}{dr} - \frac{G}{r} \right) &= 0, \\ \longrightarrow \frac{d}{dr} \left(\frac{G}{r} \frac{dN}{dr} \right) &= 0, \end{aligned} \quad (4)$$

which will be extensively used throughout this paper.

II. 2+1 PERFECT FLUID SOLUTION WITH VARIABLE $\rho(r)$

In this section, we derive the most general static circularly symmetric solution via a straightforward integration of the Einstein equations with λ for a perfect fluid. It is easy to establish that the structural functions $G(r)$ and $N(r)$ can be integrated in quadratures.

Integrating the G_{tt} -Eq. (3), one arrives at

$$G(r)^2 = -\lambda r^2 - 2\kappa \int_0^r r \rho(r) dr \equiv C - \lambda r^2 - 2\kappa \int_0^r r \rho(r) dr, \quad (5)$$

where C is an integration constant in which we have incorporated the constant value of the integral at the lower integration limit $r = 0$, thus the remaining integral depends on the upper integration limit r ; we use the r -notation for the upper integration limit as well as to denote the integration variable. This convention will be used hereafter. From the second relation of Eq. (4), one obtains

$$\frac{dN}{dr} = n_1 \frac{r}{G(r)}, \quad (6)$$

therefore

$$N(r) = n_1 \int_0^r \frac{r}{G(r)} dr \equiv n_0 + n_1 \int_0^r \frac{r}{G(r)} dr. \quad (7)$$

The evaluation of the pressure $p(r)$ yields

$$\kappa p(r) = \frac{1}{N(r)} [n_1 G(r) + \lambda N(r)]. \quad (8)$$

The metric (1), with $G(r)$ from Eq. (5), and $N(r)$ from Eq. (7), determines the general static circularly symmetric 2+1 solution of the Einstein equations (3) with λ , positive or negative, for a perfect fluid, characterized by a pressure given by Eq. (8), and an arbitrary density $\rho(r)$. The fluid-velocity is aligned along the time-like Killing direction ∂_t . In the derivation of the obtained solutions no positivity conditions were imposed, thus these fluids allow for negative p and ρ . Nevertheless, to deal with realistic matter distributions one has to impose positivity conditions on the density, $\rho > 0$, and the pressure, $p > 0$, requiring additionally $\rho > p$.

For a finite distributed fluid, the pressure p becomes zero at the boundary, say $r = a$; this value of the radial coordinate r is determined as solution of the equation $p(r) = 0$.

For non-vanishing cosmological constant, assuming that the values of the structural functions at the boundary $r = a$ are $N(a)$ and $G(a)$, the vanishing at $r = a$ of the pressure $p(r)$, given by Eq. (8), requires $n_1 = -\lambda N(a)/G(a)$, hence

$$\kappa p(r) = \frac{\lambda}{N(r)G(a)} [N(r)G(a) - N(a)G(r)]. \quad (9)$$

If one is interested in matching the obtained perfect fluid metric with a vacuum metric with cosmological constant λ , the plausible choice at hand is the anti-de Sitter metric, with $\lambda = -1/l^2$, see Sec. IV, for which $G(a) = N(a) = \sqrt{-M_\infty + a^2/l^2}$ at the boundary $r = a$. Incidentally, for a non-zero cosmological constant, there is no room for dust. The zero character of the pressure yields the vanishing of the density, and consequently the metric describes the (anti)-de Sitter spacetime.

For vanishing cosmological constant, the expression of the pressure (8) is

$$\kappa p(r) = n_1 \frac{G(r)}{N(r)}, \quad (10)$$

from which it becomes apparent that the corresponding solution represents a cosmological spacetime; there is no surface of vanishing pressure.

For vanishing λ and zero pressure, the situation slightly changes: the function N becomes a constant, and the corresponding metric can be written as

$$ds^2 = -dt^2 + \frac{dr^2}{C - 2\kappa \int_0^r r \rho(r) dr} + r^2 d\theta^2, \quad (11)$$

for any density function ρ . Of course, the choice of ρ is restricted by physically reasonable matter distributions.

III. CANONICAL COORDINATE SYSTEM $\{t, N, \theta\}$

In this section we show that an alternative formulation of our general solution can be achieved in coordinates $\{t, N, \theta\}$. Indeed, from Eq. (6) for the derivative of the function N , in which we are including—without any loss of generality—the constant n_1 , ($N/n_1 \rightarrow N, n_1 t \rightarrow t$), one obtains

$$\frac{dN}{r} = \frac{dr}{G(r)}, \quad (12)$$

hence

$$r^2 = C_0 + 2 \int^N G dN. \quad (13)$$

To derive G as a function of the new variable N , one uses the G_{tt} —Eq. (3) in the form of

$$G dG = -(\kappa \rho + \lambda) r dr = -(\kappa \rho + \lambda) G dN, \quad (14)$$

therefore, one gets

$$G(N) = C_1 - \lambda N - \kappa \int^N \rho(N) dN. \quad (15)$$

Substituting this function G into the expression (13) for r , one has

$$H(N) := r^2 = C_0 + 2 C_1 N - \lambda N^2 - 2 \kappa \int^N \int^N \rho(N) dN dN. \quad (16)$$

Finally, our metric in the new coordinates $\{t, N, \theta\}$ amounts to

$$ds^2 = -N^2 dt^2 + \frac{dN^2}{H(N)} + H(N) d\theta^2, \quad (17)$$

which is characterized by pressure

$$p(N) = \frac{C_1}{\kappa} \frac{1}{N} - \frac{1}{N} \int^N \rho(N) dN, \quad (18)$$

and an arbitrary energy density $\rho(N)$ depending on the variable N ; for physically conceivable solutions, both functions p and ρ have to be positive.

The metric (17) together with the function H from (16) give an alternative representation of our general solution. This representation will be used to derive particular solutions for a given state equation of the form $p = p(\rho)$. In this approach the expression of the pressure (18) plays a central role.

For completeness and checking purposes we include the Einstein equations for perfect fluid and cosmological constant for metric (17), considering the function H as an

arbitrary one. There are only two equations governing the problem namely:

$$G_{tt} : \frac{d^2 H}{dN^2} = -2\lambda - 2\kappa \rho(N) \quad (19)$$

$$G_{NN} : \frac{dH}{dN} = -2\lambda N + 2\kappa N p(N). \quad (20)$$

It is well known that the vanishing of the Cotton tensor in three dimensions determines locally the class of conformally flat spaces, in the same fashion as the vanishing of the Weyl tensor singles out conformally flat spaces in higher dimensions. The evaluation of the Cotton tensor

$$C_{\mu\nu\sigma} := -L_{\mu[\nu;\sigma]}, \quad L_{\mu\nu;\sigma} := R_{\mu\nu;\sigma} - \frac{1}{4} g_{\mu\nu} R_{;\sigma}, \quad (21)$$

yields only two independent non-vanishing components

$$\begin{aligned} C_{ttN} &= -\frac{1}{4} \kappa N^2 \frac{d\rho}{dN}, \\ C_{\phi\phi N} &= -\frac{1}{4} \kappa g_{\phi\phi} \frac{d\rho}{dN}. \end{aligned} \quad (22)$$

Therefore, from the above relations, we conclude that the perfect fluid solution with $\rho = \text{const.}$ for static circularly symmetric spacetimes is unique. This result can be stated as a theorem: The perfect fluid solution with constant ρ is the only conformally flat static circularly symmetric spacetime for a perfect fluid source with or without cosmological constant. It is noteworthy the similarity of this theorem with the corresponding one formulated for the interior Schwarzschild metric, see [16, 17]. We shall return to this spacetime in Sec.V.

A. 2+1 perfect fluid solutions for a linear law

$$p = \gamma \rho$$

Although in the previous section we provided the general solution to the posed question of finding all solutions for circularly symmetric static metric in 2+1 gravity coupled to perfect fluid in the presence of the cosmological constant, from the physical point of view, even in this lower dimensional spacetime, it is of interest to analyze certain specific cases, for instance, the solution corresponding to a fluid obeying the law $p = \gamma \rho$, or the more complicated case of a polytropic law $p = C \rho^\gamma$.

The starting point in the present study is the linear relation between pressure and energy density

$$p(N) = \gamma \rho(N). \quad (23)$$

Substituting $p(N)$ from Eq. (18) into this relation, one gets

$$\frac{C_1}{\kappa} - \int^N \rho(N) dN = \gamma N \rho(N). \quad (24)$$

Differentiating this equation with respect to the variable N , one obtains

$$\frac{d}{dN}(N \rho) + \frac{1}{\gamma N}(N \rho) = 0, \quad (25)$$

which has as general integral

$$\rho(N) = C_2 \frac{\gamma-1}{\gamma^2} N^{-\frac{\gamma+1}{\gamma}}, \quad (26)$$

where C_2 is an integration constant. Since we arrived at $\rho(N)$, Eq. (26), through differentiation, then one has to replace the obtained result into the relation (24), or equivalently into Eq. (23), to see if there arises any constraint from it:

$$p(N) = \frac{C_1}{\kappa} \frac{1}{N} + \gamma \rho(N) = \gamma \rho(N) \longrightarrow C_1 = 0, \quad (27)$$

In such manner, we have established that the constant C_1 vanishes. Replacing the function $\rho(N)$ from Eq. (26) into the expression of $H(N)$, Eq. (16), and accomplishing the integration one arrives at

$$H(N) = C_0 - \lambda N^2 + 2\kappa C_2 N^{(\gamma-1)/\gamma} = r^2. \quad (28)$$

Thus, the metric for a perfect fluid fulfilling a linear state equation in coordinates $\{t, N, \theta\}$ is given by

$$ds^2 = - N^2 dt^2 + \frac{dN^2}{C_0 - \lambda N^2 + 2\kappa C_2 N^{(\gamma-1)/\gamma}} + (C_0 - \lambda N^2 + 2\kappa C_2 N^{(\gamma-1)/\gamma}) d\theta^2. \quad (29)$$

To express this solution in terms of the radial variable r , one has to be able to solve the algebraic equation Eq. (28), in general a transcendent one, for $N = N(r)$.

The evaluation of the Cotton tensor leads to two independent non-vanishing components:

$$\begin{aligned} C_{ttN} &= C_2 \frac{\kappa(\gamma^2-1)}{4\gamma^3} N^{-\frac{1}{\gamma}}, \\ C_{\phi\phi N} &= C_2 \frac{\kappa(\gamma^2-1)}{4\gamma^3} N^{-\frac{2\gamma+1}{\gamma}} g_{\phi\phi}. \end{aligned} \quad (30)$$

Some examples of physical interest are described by the treated state equation, for instance: dust, $\gamma = 0$, stiff matter, $\gamma = 1$, pure radiation, $\gamma = 1/2$, and incoherent radiation, $\gamma = 1/3$. Details the reader may encounter in [18].

B. 2+1 perfect fluid solutions for a polytropic law

$p = C\rho^\gamma$

This subsection is devoted to the derivation of all solutions obeying the polytropic law

$$p = C\rho^\gamma. \quad (31)$$

Using again the expression of $p(N)$ from Eq. (18), the above polytropic relation can be written as

$$\frac{C_1}{\kappa} - \int^N \rho(N) dN = C N \rho^\gamma(N). \quad (32)$$

Differentiating with respect to N , one obtains

$$-\rho = C \frac{d}{dN} (N \rho^\gamma), \quad (33)$$

which, by introducing the auxiliary function $Z = N^{1/\gamma} \rho$, can be written as

$$d(Z^{\gamma-1}) + \frac{1}{C} d(N^{(\gamma-1)/\gamma}) = 0, \quad (34)$$

therefore

$$d \left[\left(\rho^{\gamma-1} + \frac{1}{C} \right) N^{(\gamma-1)/\gamma} \right] = 0. \quad (35)$$

Consequently, the general integral of the studied equation becomes

$$\rho = C^{-\frac{1}{\gamma}} N^{-\frac{1}{\gamma}} \left[B - C^{-\frac{1}{\gamma}} N^{\frac{\gamma-1}{\gamma}} \right]^{\frac{1}{\gamma-1}}, \quad (36)$$

where B is an integration constant. Entering this ρ into the equation (31), taking into account that the integral of the density ρ amounts to

$$\int^N \rho(N) dN = - \int^N d \left[B - C^{-\frac{1}{\gamma}} N^{\frac{\gamma-1}{\gamma}} \right]^{\frac{\gamma}{\gamma-1}}, \quad (37)$$

one arrives at

$$p(N) = \frac{n_1}{\kappa} \frac{C_1}{N} + C \rho^\gamma = C \rho^\gamma \longrightarrow C_1 = 0. \quad (38)$$

Considering that the first integral of ρ is given by (37), the expression of the structural function $H(N)$ becomes

$$H = C_0 - \lambda N^2 + 2\kappa \int^N \left[B - C^{-\frac{1}{\gamma}} N^{\frac{\gamma-1}{\gamma}} \right]^{\frac{\gamma}{\gamma-1}} dN = r^2. \quad (39)$$

Notice that the mentioned integral can be expressed in terms of hypergeometric functions, hence

$$H(N) = C_0 - \lambda N^2 + 2\kappa B^{\gamma/(\gamma-1)} N F \left(\left[\frac{\gamma}{\gamma-1}, -\frac{\gamma}{\gamma-1} \right], \left[\frac{\gamma}{\gamma-1} + 1 \right], N^{(\gamma-1)/\gamma} C^{-1/\gamma} B^{-1} \right). \quad (40)$$

Summarizing, in the case of a polytropic equation of state, the general solution is given by the metric (17) with $H(N)$ from (39), and is characterized by energy density and pressure of the form:

$$\begin{aligned} p &= \frac{1}{N} \left[B - C^{-\frac{1}{\gamma}} N^{\frac{\gamma-1}{\gamma}} \right]^{\frac{\gamma}{\gamma-1}}, \\ \rho &= C^{-\frac{1}{\gamma}} N^{-\frac{1}{\gamma}} \left[B - C^{-\frac{1}{\gamma}} N^{\frac{\gamma-1}{\gamma}} \right]^{\frac{1}{\gamma-1}}. \end{aligned} \quad (41)$$

Incidentally, for zero λ , the derivation and study of static circularly symmetric cosmological spacetimes, coupled to perfect fluids fulfilling the polytropic law was accomplished in [13], where also have been discussed Robertson—Walker cosmologies.

The non—vanishing independent components of the Cotton tensor (21) are

$$\begin{aligned} C_{ttN} &= \frac{1}{4} \kappa N^2 \frac{d^2}{dN^2} \left[C_1 - C^{-\frac{1}{\gamma}} N^{\frac{\gamma-1}{\gamma}} \right]^{\frac{\gamma}{\gamma-1}}, \\ C_{\phi\phi N} &= \frac{1}{4} \kappa g_{\phi\phi} \frac{d^2}{dN^2} \left[C_1 - C^{-\frac{1}{\gamma}} N^{\frac{\gamma-1}{\gamma}} \right]^{\frac{\gamma}{\gamma-1}}. \end{aligned} \quad (42)$$

These polytropic fluids contain, amongst others, certain physically relevant samples: nonrelativistic degenerate fermions, $\gamma = 2$, nonrelativistic matter, $\gamma = 3/2$, monatomic and diatomic gases, $\gamma = 7/5$ and $\gamma = 5/3$ respectively.

IV. OPPENHEIMER—VOLKOV EQUATION

Although when Einstein equations have been fulfilled, the energy-momentum conservation law trivially holds, it is of interest to establish certain properties arising from the Oppenheimer—Volkov equation, see for instance [14] in 2+1 gravity. An alternative derivation of this equation consists in differentiating with respect to r the Einstein G_{rr} -Eq. (3), this yields

$$\begin{aligned} \kappa \frac{dp}{dr} &= \frac{1}{rN} \frac{dN}{dr} \left(\frac{dG^2}{dr} - \frac{G^2}{r} \right) \\ &+ \frac{G^2}{rN} \left(\frac{d^2N}{dr^2} - \frac{1}{N} \left(\frac{dN}{dr} \right)^2 \right). \end{aligned} \quad (43)$$

Substituting the second derivative d^2N/dr^2 from Eq. (4), and the first derivative dN/dr from the G_{rr} -equation into Eq. (43), one arrives at the Oppenheimer—Volkov equation:

$$\frac{dp}{dr} = -\frac{r}{G^2} (\kappa p - \lambda) (\rho + p). \quad (44)$$

At the circle $r = a$ of vanishing pressure $p(a) = 0$, the pressure gradient amounts to

$$\left. \frac{dp}{dr} \right|_{r=a} = \frac{\lambda a}{G(a)^2} \rho(a). \quad (45)$$

Since inside the circle the pressure is positive, $p(r < a) > 0$, hence at the circle $r = a$ the pressure gradient has to

be non-positive, consequently the cosmological constant ought to be negative, $\lambda = -1/l^2 < 0$. We shall continue to use λ instead of $-1/l^2$, keeping in mind that λ is a negative constant.

The definition of the mass contained in the circle of radius a is given by

$$M := 2\pi \int_0^a \rho(r) r dr, \quad (46)$$

and since the metric components $g_{rr} = 1/G(r)^2$ has to be positive in the domain of definition of the solution, then there exists an upper limit for the mass, namely

$$M \leq \frac{\pi}{\kappa} (C - \lambda a^2). \quad (47)$$

Assuming that a state equation $p = p(\rho)$ holds, the matter content is said to be microscopically stable if $dp/d\rho \geq 0$. Since Eq. (44) can be written as

$$\frac{dp}{d\rho} = -\frac{r}{G^2} (\kappa p - \lambda) (\rho + p) / \frac{d\rho}{dr}, \quad (48)$$

one concludes that for a microscopically stable fluid with positive pressure p and positive density ρ , this density occurs to be monotonically decreasing $d\rho/dr < 0$. Moreover, the physical requirement that the sound speed is less than the velocity of light imposes an upper limit on $dp/d\rho \leq 1$.

For our general solution in coordinates $\{t, N, \theta\}$, metric (17), from the expression (18) for the pressure, one establishes

$$\frac{dp}{d\rho} = -\frac{1}{N} (\rho + p) / \frac{d\rho}{dN}, \quad (49)$$

therefore the density is monotonically decreasing $d\rho/dN < 0$ if the matter is microscopically stable $dp/d\rho \geq 0$, and conversely.

Moreover, our fluids, fulfilling the state equation $p = \gamma\rho$, $\gamma > 0$, as well as those ones obeying the polytropic law $p = C\rho^\gamma$, $C > 0, \gamma > 0$, are microscopically stable fluids.

V. 2+1 PERFECT FLUID SOLUTION WITH CONSTANT ρ

As we demonstrated in Sec. II, for constant ρ the Cotton tensor vanishes, and consequently the corresponding conformally flat space occurs to be unique.

In this section it is shown that one can achieve a full correspondence of the metrics and structural functions for constant energy density perfect fluids in 2+1 and 3+1 gravities. By an appropriate choice of the constant densities and cosmological constants, via a dimensional reduction (freezing of one of the spatial coordinates of the 3+1 spacetime), one obtains the 2+1 metric structure from the 3+1 solution. To achieve the mentioned purpose, the

conformally flat static spherically symmetric perfect fluid 3+1 solution with cosmological constant is presented in a form which allows a comparison with the static circularly symmetric perfect fluid with λ -term and constant ρ of the 2+1 gravity.

In the canonical coordinate system $\{t, N, \theta\}$, for $\rho = \text{const.}$, the metric, the expression of the function H , which on its turn establishes the relation to the radial coordinate r , and the pressure are given by:

$$ds^2 = -N^2 dt^2 + \frac{dN^2}{H} + H d\theta^2, \quad (50)$$

$$H = C_0 + 2C_1 N - (\lambda + \kappa\rho_0)N^2 = r^2, \quad (51)$$

$$p = -\rho_0 + \frac{C_1}{\kappa} \frac{1}{N}. \quad (52)$$

This unfamiliar looking solution can be given in terms of the radial variable r by expressing N as function of r , $N = N(r)$.

Having in mind the comparison of the 2+1 constant ρ perfect fluid with its 3+1 relative—the Schwarzschild interior solution—we shall derive it from the very beginning by integrating the Einstein equations (3) in coordinates $\{t, r, \theta\}$.

For $\rho = \text{const.}$, the integral of (5) gives

$$G(r) = \sqrt{C - (\kappa\rho + \lambda)r^2}. \quad (53)$$

Substituting $G(r)$ from (53) into (7), one obtains

$$N(r) = n_0 - \frac{n_1}{\lambda + \kappa\rho} G(r), \quad (54)$$

which can be written as $N(r) = C_1 + C_2 G(r)$.

The evaluation of pressure $p(r)$ from Eq. (3) yields

$$\kappa p(r) = \frac{1}{(\kappa\rho + \lambda)N(r)} [n_1 \kappa\rho G(r) + n_0 \lambda(\kappa\rho + \lambda)], \quad (55)$$

this pressure has to vanish at the boundary $r = a$, which imposes a relation on the constants: $n_0 = -n_1 \kappa\rho G(a)/[\lambda(\kappa\rho + \lambda)]$, where $G(a)$ is the value of the function $G(r)$ at the boundary, i.e., $G(a)$ is equal to the external value for the $G(r)$ corresponding to the vacuum solution plus λ . A similar comment applies to $N(a)$. Replacing n_0 in Eq. (54), the function $N(r)$ becomes

$$N(r) = -\frac{n_1}{\lambda(\kappa\rho + \lambda)} [\kappa\rho G(a) + \lambda G(r)]. \quad (56)$$

Evaluating $N(r)$ at $r = a$, one comes to $n_1 = -\lambda N(a)/G(a)$. Consequently, $N(r)$ amounts to

$$N(r) = \frac{N(a)}{G(a)(\kappa\rho + \lambda)} [\kappa\rho G(a) + \lambda G(r)]. \quad (57)$$

Substituting n_0 , n_1 , and $N(r)$ into (55), one gets

$$p(r) = \rho \lambda \frac{G(a) - G(r)}{\kappa\rho G(a) + \lambda G(r)}. \quad (58)$$

Summarizing, the studied perfect fluid for the metric (1) is determined by structural functions $G(r)$ from Eq. (53), and $N(r)$ from Eq. (57), and characterized by a density $\rho = \text{const.}$, and pressure p given by Eq. (58).

The $g_{tt} = -N^2$ metric component, with N from (57), corrects the corresponding one, reported in [14].

A. 3+1 conformally flat static spherically symmetric perfect fluid solution

In this subsection we review the main structure of the interior perfect fluid solution in the presence of the cosmological constant λ —the interior Schwarzschild metric with λ —for the 3 + 1 static spherically symmetric metric of the form

$$ds^2 = -N(r)^2 dt^2 + \frac{dr^2}{G(r)^2} + r^2 (d\theta^2 + \sin^2\theta d\phi^2). \quad (59)$$

The Einstein equations for a perfect fluid energy-momentum tensor in four dimensions have the same form of the ones in three dimensions (2), except for modifications due to the change of dimensionality, namely, the expression of R now amounts to $R = -\kappa T + 4\lambda = \kappa\rho - 3\kappa p + 4\lambda$. Because the corresponding equations can be found in text-books, we do not exhibit them here explicitly; we include them for reference in the Appendix.

Since we are interested in conformally flat solutions, we require the vanishing of the conformal Weyl tensor, which for static spherically symmetric spacetime leads to the following equation

$$\frac{d}{dr} \left(\frac{G^2 - 1}{r^2} \right) = 0 \longrightarrow G(r) = \sqrt{1 + c_0 r^2}. \quad (60)$$

On the other hand, from the G_{tt} -Eq., one arrives at

$$G(r) = \sqrt{1 - \frac{1}{3}(\kappa\rho + \lambda)r^2}, \quad (61)$$

therefore, comparing with Eq. (60), one has $c_0 = -(\kappa\rho + \lambda)/3$, $\longrightarrow \rho = \text{const.}$ Hence, the solution constructed under this condition would correspond to a perfect fluid with $\rho = \text{const.}$, named incompressible fluid by Adler *et al.* [19].

Moreover, from the Eq. $(r^2 G^2 G_{rr} - G_{\theta\theta}) = 0$, taking into account the form of the function G from (61), the general expression of $N(r)$ is

$$N(r) = C_1 + C_2 G(r). \quad (62)$$

The evaluation of the pressure p , from G_{rr} -Eq., yields

$$\kappa p(r) = \frac{1}{3N(r)} [C_1 (2\lambda - \kappa\rho) - 3C_2 \kappa\rho G(r)]. \quad (63)$$

where $G(r)$ and $N(r)$ are determined in (61), and (62) respectively.

This result can be stated in the form of a generalization

of the Gürses and Gürsey theorem [20] to the case of non-zero λ : the only conformally flat spherically symmetric static solution to the Einstein equations with cosmological constant for a perfect fluid is given by the metric (59) with structural functions $G(r)$ and $N(r)$ defined respectively by (61) and (62). Moreover, replacing in the metric (59) $\sin^2\theta$ by $\sinh^2\theta$, and θ^2 , one obtains correspondingly the pseudospherical and flat branches of solutions.

The constants C_1 and C_2 are determined through the values of structural functions at the boundary $r = a$, where the pressure vanishes, $p(r = a) = 0$, they occur to be:

$$C_1 = 3 C_2 \kappa \rho \frac{G(a)}{2\lambda - \kappa \rho}, \quad C_2 = \frac{N(a)}{2G(a)} \frac{2\lambda - \kappa \rho}{\lambda + \kappa \rho}, \quad (64)$$

where $G(a)$ is the value of the function $G(r)$ at the boundary $r = a$, i.e., $G(a)$ is equal to the external value of $G(r)$ corresponding to the vacuum plus λ solution. A similar comment applies to $N(a)$. We shall return to this point at the end of this subsection.

Substituting the expressions of C_1 and C_2 into Eq. (62), one has

$$N(r) = \frac{N(a)}{2G(a)(\kappa\rho + \lambda)} [3\kappa\rho G(a) + (2\lambda - \kappa\rho)G(r)]. \quad (65)$$

Replacing C_1 , C_2 and the above expression of $N(r)$ into (63), one gets

$$p(r) = \rho(2\lambda - \kappa\rho) \frac{G(a) - G(r)}{3\kappa\rho G(a) + (2\lambda - \kappa\rho)G(r)}. \quad (66)$$

For the external Schwarzschild with λ solution, known also the Kottler solution [21], the functions $N(r)$ and $G(r)$ are equal one to another, $N(r) = G(r)$, namely

$$N(r) = G(r) = \sqrt{1 - \frac{2m}{r} - \frac{\lambda}{3}r^2}, \quad \text{for } r \geq a. \quad (67)$$

Evaluating the mass contained in the sphere of radius a for a constant density ρ , one obtains $2m = \kappa\rho a^3/3$, therefore

$$N(r) = G(r) = \sqrt{1 - \frac{\kappa\rho a^3}{3r} - \frac{\lambda}{3}r^2}, \quad \text{for } r \geq a, \quad (68)$$

consequently at $r = a$, one has

$$N(a) = G(a) = \sqrt{1 - \frac{\kappa\rho + \lambda}{3}a^2}. \quad (69)$$

In the limit of vanishing cosmological constant, $\lambda = 0$, one arrives at the interior Schwarzschild solution.

B. Comparison table

A comparison table of perfect fluid solutions with constant ρ in 2+1 and 3+1 gravities is given as:

Perfect fluid solutions with constant energy density

3 + 1 – solution	2 + 1 – solution
$ds^2 = -N^2 dt^2 + \frac{dr^2}{G^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$	$ds^2 = -N^2 dt^2 + \frac{dr^2}{G^2} + r^2 d\phi^2$
$G^2 = 1 - \frac{1}{3}(\kappa\rho + \lambda)r^2$	$G^2 = C - (\kappa\rho + \lambda)r^2$
$N = \frac{1}{2(\kappa\rho + \lambda)} \frac{N(a)}{G(a)} [3\kappa\rho G(a) + (2\lambda - \kappa\rho)G(r)]$	$N = \frac{1}{(\kappa\rho + \lambda)} \frac{N(a)}{G(a)} [\kappa\rho G(a) + \lambda G(r)]$
$p(r) = \rho(2\lambda - \kappa\rho) \frac{G(a) - G(r)}{3\kappa\rho G(a) + (2\lambda - \kappa\rho)G(r)}$	$p(r) = \rho\lambda \frac{G(a) - G(r)}{\kappa\rho G(a) + \lambda G(r)}$
Kottler :	anti-de Sitter: $\lambda = -1/l^2$
$G(a)^2 = 1 - \frac{2m}{r} - \frac{1}{3}\lambda r^2;$ $2m = \kappa\rho a^3/3$	$G(a)^2 = -M_\infty - \lambda a^2;$ $C = \kappa\rho a^2 - M_\infty > 0$
$N(a) = G(a)$	$N(a) = G(a)$
$2\lambda_4 - \kappa_4\rho_4 \rightarrow 6\lambda_3, \kappa_4\rho_4 \rightarrow 2\kappa_3\rho_3$	
$G_4(r) \rightarrow G_3(r), N_4(r) \rightarrow N_3(r), \kappa_4 p_4(r) \rightarrow 2\kappa_3 p_3(r)$	

Comparing the structure corresponding to perfect fluid solutions with constant ρ in 3+1 gravity with the structure of the 2+1 perfect fluid solution one arrives at the following correspondence : $2\lambda_4 - \kappa_4\rho_4 \rightarrow 6\lambda_3$, $3\kappa_4\rho_4 \rightarrow 6\kappa_3\rho_3$, which yields $G_4(r) \rightarrow G_3(r)$, $N_4(r) \rightarrow N_3(r)$,

$\kappa_4 p_4(r) \rightarrow 2\kappa_3 p_3(r)$. Remembering that in 2+1 gravity there is no Newtonian limit, the choice of κ_3 is free, thus by selecting κ_3 appropriately one can achieve that $p_4(r) \rightarrow p_3(r)$ and $\rho_4 \rightarrow \rho_3$.

From this comparison table one easily conclude that the 2+1 perfect fluid with constant ρ can be derived from the Schwarzschild interior metric by a simple dimensional reduction: freezing one of the spatial coordinates, say $\theta = \pi/2$, in the 3+1 solution one obtains the corresponding 2+1 spacetime.

Since we accomplished a scaling transformation of the r -coordinate, accompanied with the inverse scaling of the angular coordinate ϕ , one may argue that a conical singularity could arise; one may overcome this trouble by saying that the angular coordinate should be fixed once one brings the 2+1 metric to the canonical form with $G_3(r) = \sqrt{1 - (\kappa\rho + \lambda)r^2}$.

VI. CONCLUDING REMARKS

In this contribution we have derived all perfect fluid solutions for the static circularly symmetric spacetime. The general solution is presented in the standard coordinate system $\{t, r, \theta\}$, and alternatively, in a system—the canonical one—with coordinates $\{t, N, \theta\}$. From the physical point of view, particularly interesting are those fluids fulfilling the linear equation of state, $p = \gamma\rho$, as well as those ones subjected to the polytropic law $p = C\rho^\gamma$; both families are derived in details from our general metric referred to the coordinate system $\{t, N, \theta\}$. Therefore, the derived solutions describe, among others, stiff matter, pure radiation, incoherent radiation, nonrelativistic degenerate fermions, etc. The constant energy density perfect fluid solution with cosmological constant of the 2+1 gravity is singled out among all static circularly spacetimes as the

only conformally flat space—its Cotton tensor vanishes—sharing the conformally flatness property with its 3+1 counterpart—the Schwarzschild interior perfect fluid solution with λ ; a comparison table for these solutions with constant energy density is included.

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APPENDIX A: 3+1 EINSTEIN EQUATIONS WITH λ FOR PERFECT FLUID

The Einstein equations with cosmological constant for perfect fluids for the metric (59) explicitly amount to

$$\begin{aligned} G_{tt} &= -\frac{N^2}{r^2} \left(r \frac{dG^2}{dr} + G^2 - 1 \right) = N^2(\kappa\rho + \lambda), \\ G_{rr} &= \frac{1}{G^2 N r^2} \left(2rG^2 \frac{dN}{dr} - N + NG^2 \right) = \frac{1}{G^2}(\kappa p - \lambda), \\ G_{\theta\theta} &= \frac{r}{N} \left(G^2 \frac{dN}{dr} + \frac{1}{2} N \frac{dG^2}{dr} + rG^2 \frac{d^2 N}{dr^2} + \frac{r}{2} \frac{dN}{dr} \frac{dG^2}{dr} \right) \\ &= r^2(\kappa p - \Lambda), \\ G_{\phi\phi} &= \sin^2\theta \ G_{\theta\theta}. \end{aligned} \tag{A1}$$

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